Second order differential equations with constant coefficients

The general form of linear Second order differential equations with constant coefficients is

$(aD^2 + bD + c)\, y = X \longrightarrow$ (i)

Where a,b,c are constants and X is a function of x.and D = $\dfrac{d}{dx}$

When X is equal to zero, then the equation is said to be homogeneous.

Let D = m Then equation (i) becomes

$am^2 + bm + c = 0$

This is known as **auxiliary equation**. This quadratic equation has two roots say m_1 and m_2.

The solution consists of one part namely complementary function

(ie) y = complementary function

Complementary Function

Case (i)

If the roots (m_1 & m_2) are real and distinct ,then the solution is given by

$y = Ae^{m_1 x} + Be^{m_2 x}$ where A and B are the two arbitrary constants.

Case (ii)

If the roots are equal say $m_1 = m_2 = m$, then the solution is given by

$y = (A + Bx)e^{mx}$ where A and B are the two arbitrary constants.

Case (iii)

If the roots are imaginary say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$

Where α and β are real. The solution is given by $y = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$ where A and B are arbitrary constants.

Particular integral

The equation $(aD^2 + bD + c)y = X$ is called a non homogeneous second order linear equation with constant coefficients. Its solution consists of two terms complementary function and particular Integral.

(ie) y = complementary function + particular Integral

Let the given equation is f(D) y(x) = X

$$y(x) = \frac{X}{f(D)}$$

Case (i)

Let X= $e^{\alpha x}$ and f(α) $\neq 0$

Then P.I = $\frac{1}{f(D)} e^{\alpha x} = \frac{1}{f(\alpha)} e^{\alpha x}$

Case (ii)

Let X = P(x) where P(x) is a polynomial

Then P.I = $\frac{1}{f(D)}$P(x) = $[f(D)]^{-1}$ P(x)

Write $[f(D)]^{-1}$ in the form $(1 \pm x)^{-1}$ $(1 \pm x)^{-2}$ and proceed to find higher order derivatives depending on the degree of the polynomial.

Newton's Law of Cooling

Rate of change in the temperature of an object is proportional to the difference between the temperature of the object and the temperature of an environment. This is known as Newton's law of cooling. Thus, if θ is the temperature of the object at time *t*, then we have

$$\frac{d\theta}{dt} \ \alpha \ \theta$$

$$\frac{d\theta}{dt} = -k(\theta)$$

This is a first order linear differential equation.

Population Growth

The differential equation describing exponential growth is

$$\frac{dG}{dt} = KG$$

This equation is called the law of growth, and the quantity K in this equation is sometimes known as the Malthusian parameter.