CHAIN RULE DIFFERENTIATION

If y is a function of u ie y = f(u) and u is a function of x ie u = g(x) then y is related to x through the intermediate function u ie y =f(g(x))

$\therefore$ y is differentiable with respect to x

Furthermore, let y=f(g(x)) and u=g(x), then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

There are a number of related results that also go under the name of "chain rules." For example, if y=f(u) u=g(v), and v=h(x),

then $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Problem

Differentiate the following with respect to x

1. $y = (3x^2+4)^3$
2. $y = e^{x^{-2}}$

Marginal Analysis

Let us assume that the total cost C is represented as a function total output q. (i.e) C= f(q).

Then marginal cost is denoted by MC= $\frac{dc}{dq}$

The average cost = $\frac{TC}{Q}$

Similarly if U = u(x) is the utility function of the commodity x then

the marginal utility MU = $\frac{dU}{dx}$

The total revenue function TR is the product of quantity demanded Q and the price P per unit of that commodity then TR = Q.P = f(Q)

Then the marginal revenue denoted by MR is given by $\frac{dR}{dQ}$

The average revenue = $\frac{TR}{Q}$

Problem

1. If the total cost function is $C = Q^3 - 3Q^2 + 15Q$. Find Marginal cost and average cost.

Solution

MC = $\frac{dc}{dq}$

AC = $\frac{TC}{Q}$

2. The demand function for a commodity is P= (a - bQ). Find marginal revenue.

(the demand function is generally known as Average revenue function). Total revenue TR = P.Q = Q. (a - bQ) and marginal revenue MR= $\frac{d\left(aQ-bQ^2\right)}{dq}$

Growth rate and relative growth rate

The growth of the plant is usually measured in terms of dry mater production and as denoted by W. Growth is a function of time t and is denoted by W=g(t) it is called a growth function. Here t is the independent variable and w is the dependent variable.

The derivative $\frac{dw}{dt}$ is the growth rate (or) the absolute growth rate gr= $\frac{dw}{dt}$. GR = $\frac{dw}{dt}$

The relative growth rate i.e defined as the absolute growth rate divided by the total dry matter production and is denoted by RGR.

i.e RGR = $\frac{1}{w} \cdot \frac{dw}{dt} = \frac{absolute\,growth\,rate}{total\,dry\,matter\;production}$

Problem

1. If G = at^2+b sin t +5 is the growth function function the growth rate and relative growth rate.

   GR = $\frac{dG}{dt}$

   RGR = $\frac{1}{G} \cdot \frac{dG}{dt}$

Implicit Functions

If the variables x and y are related with each other such that f (x, y) = 0 then it is called Implicit function. A function is said to be **explicit** when one variable can be expressed completely in terms of the other variable.

For example, $y = x^3 + 2x^2 + 3x + 1$ is an Explicit function

$xy^2 + 2y + x = 0$ is an implicit function

Problem

For example, the implicit equation xy=1 can be solved by differentiating implicitly gives

$$\frac{d(xy)}{dx} = \frac{d(1)}{dx}$$

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}.$$

Implicit differentiation is especially useful when y'(x)is needed, but it is difficult or inconvenient to solve for y in terms of x.

Example: *Differentiate the following function with respect to x* $x^3 y^6 + \mathbf{e}^{1-x} - \cos(5y) = y^2$

Solution

So, just differentiate as normal and tack on an appropriate derivative at each step. Note as well that the first term will be a product rule.

$$3x^2 x' y^6 + 6x^3 y^5 y' - x' \mathbf{e}^{1-x} + 5y' \sin(5y) = 2yy'$$

Example: Find y' for the following function.

$$x^2 + y^2 = 9$$

Solution

In this example we really are going to need to do implicit differentiation of *x* and write *y* as *y(x)*.

$$\frac{d}{dx}\left(x^2 + \left[y(x)\right]^2\right) = \frac{d}{dx}(9)$$

$$2x + 2\left[y(x)\right]^1 y'(x) = 0$$

Notice that when we differentiated the *y* term we used the chain rule.

Example: Find y' for the following. $x^3 y^5 + 3x = 8y^3 + 1$

Solutio: First differentiate both sides with respect to *x* and notice that the first time on left side will be a product rule.

$$3x^2 y^5 + 5x^3 y^4 y' + 3 = 24y^2 y'$$

Remember that very time we differentiate a *y* we also multiply that term by $y'y'$ since we are just using the chain rule. Now solve for the derivative.

$$3x^2y^5+3=24y^2y'-5x^3y^4y'$$
$$3x^2y^5+3=\left(24y^2-5x^3y^4\right)y'$$
$$y'=\frac{3x^2y^5+3}{24y^2-5x^3y^4}$$

The algebra in these can be quite messy so be careful with that.

Example: Find y' for the following $x^2\tan(y)+y^{10}\sec(x)=2x$

Here we've got two product rules to deal with this time.

$$2x\tan(y)+x^2\sec^2(y)y'+10y^9y'\sec(x)+y^{10}\sec(x)\tan(x)=2$$

Notice the derivative tacked onto the secant. We differentiated a *y* to get to that point and so we needed to tack a derivative on.

Now, solve for the derivative.

$$\left(x^2\sec^2(y)+10y^9\sec(x)\right)y'=2-y^{10}\sec(x)\tan(x)-2x\tan(y)$$
$$y'=\frac{2-y^{10}\sec(x)\tan(x)-2x\tan(y)}{x^2\sec^2(y)+10y^9\sec(x)}$$

Logarithmic Differentiation

For some problems, first by taking logarithms and then differentiating, it is easier to find $\frac{dy}{dx}$. Such process is called Logarithmic differentiation.

(i) If the function appears as a product of many simple functions then by taking logarithm so that the product is converted into a sum. It is now easier to differentiate them.

(ii) If the variable x occurs in the exponent then by taking logarithm it is reduced to a familiar form to differentiate.

Example

Differentiate the function.

$$y=\frac{x^5}{(1-10x)\sqrt{x^2+2}}$$

Solution

Differentiating this function could be done with a product rule and a quotient rule. We can simplify things somewhat by taking logarithms of both sides.

$$\ln y = \ln\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right)$$

$$\ln y = \ln\left(x^5\right) - \ln\left((1-10x)\sqrt{x^2+2}\right)$$

$$\ln y = \ln\left(x^5\right) - \ln(1-10x) - \ln\left(\sqrt{x^2+2}\right)$$

$$\frac{y'}{y} = \frac{5x^4}{x^5} - \frac{-10}{1-10x} - \frac{\frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}(2x)}{\left(x^2+1\right)^{\frac{1}{2}}}$$

$$\frac{y'}{y} = \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+1}$$

Example

Differentiate $y = x^x$

Solution

First take the logarithm of both sides as we did in the first example and use the logarithm properties to simplify things a little.

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

Differentiate both sides using implicit differentiation.

$$\frac{y'}{y} = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$$

As with the first example multiply by *y* and substitute back in for *y*.

$$y' = y(1 + \ln x)$$

$$= x^x(1 + \ln x)$$

PARAMETRIC FUNCTIONS

Sometimes variables x and y are expressed in terms of a third variable called **parameter.** We find $\frac{dy}{dx}$ without eliminating the third variable.

Let x = f(t) and y = g(t) then

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} = \frac{dy/dt}{dx/dt}$$

Problem

1. Find for the parametric function x =a cos θ , y = b sin θ

Solution

$$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{b\cos\theta}{-a\sin\theta}$$

$$= -\frac{b}{a}\cot\theta$$

Inference of the differentiation

Let y = f(x) be a given function then the first order derivative is $\frac{dy}{dx}$.

The geometrical meaning of the first order derivative is that it represents the slope of the curve y = f(x) at x.

The physical meaning of the first order derivative is that it represents the rate of change of y with respect to x.

PROBLEMS ON HIGHER ORDER DIFFERENTIATION

The rate of change of y with respect x is denoted by $\frac{dy}{dx}$ and called as the first order derivative of function y with respect to x.

The first order derivative of y with respect to x is again a function of x, which again be differentiated with respect to x and it is called second order derivative of y = f(x) and is denoted by $\frac{d^2y}{dx^2}$ which is equal to $\frac{d}{dx}\left(\frac{dy}{dx}\right)$

In the similar way higher order differentiation can be defined. Ie. The nth order derivative of y=f(x) can be obtained by differentiating n-1th derivative of y=f(x)

$$\frac{d^n y}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1} y}{dx^{n-1}}\right)$$ where n= 2,3,4,5….

Problem

Find the first , second and third derivative of

1. y = e^{ax+b}
2. y = log(a-bx)
3. y = sin (ax+b)

Partial Differentiation

So far we considered the function of a single variable y = f(x) where x is the only independent variable. When the number of independent variable exceeds one then we call it as the function of several variables.

Example

z = f(x,y) is the function of two variables x and y , where x and y are independent variables.

U=f(x,y,z) is the function of three variables x,y and z , where x, y and z are independent variables.

In all these functions there will be only one dependent variable.

Consider a function z = f(x,y). The partial derivative of z with respect to x denoted by $\frac{\partial z}{\partial x}$ and is obtained by differentiating z with respect to x keeping y as a constant. Similarly the partial derivative of z with respect to y denoted by $\frac{\partial z}{\partial y}$ and is obtained by differentiating z with respect to y keeping x as a constant.

Problem

1. Differentiate U = log (ax+by+cz) partially with respect to x, y & z

We can also find higher order partial derivatives for the function z = f(x,y) as follows

(i) The second order partial derivative of z with respect to x denoted as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$ is obtained by partially differentiating $\frac{\partial z}{\partial x}$ with respect to x. this is also known as direct second order partial derivative of z with respect to x.

(ii)The second order partial derivative of z with respect to y denoted as $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}$ is obtained by partially differentiating $\frac{\partial z}{\partial y}$ with respect to y this is also known as direct second order partial derivative of z with respect to y

(iii) The second order partial derivative of z with respect to x and then y denoted as $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial y \partial x}$ is obtained by partially differentiating $\frac{\partial z}{\partial x}$ with respect to y. this is also known as mixed second order partial derivative of z with respect to x and then y

iv) The second order partial derivative of z with respect to y and then x denoted as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x \partial y}$ is obtained by partially differentiating $\frac{\partial z}{\partial y}$ with respect to x. this is also known as mixed second order partial derivative of z with respect to y and then x. In similar way higher order partial derivatives can be found.

Problem

Find all possible first and second order partial derivatives of

1) $z = \sin(ax + by)$

2) $u = xy + yz + zx$

Homogeneous Function

A function in which each term has the same degree is called a homogeneous function.

Example

1) $x^2 - 2xy + y^2 = 0 \rightarrow$ homogeneous function of degree 2.
2) $3x + 4y = 0 \quad \rightarrow$ homogeneous function of degree 1.
3) $x^3 + 3x^2y + xy^2 - y^3 = 0 \rightarrow$ homogeneous function of degree 3.

To find the degree of a homogeneous function we proceed as follows.

Consider the function f(x,y) replace x by tx and y by ty if $f(tx, ty) = t^n f(x, y)$ then n gives the degree of the homogeneous function. This result can be extended to any number of variables.

Problem

Find the degree of the homogeneous function

1. $f(x, y) = x^2 - 2xy + y^2$

2. f(x,y) = $\dfrac{x-y}{x+y}$

Euler's theorem on homogeneous function

If U= f(x,y,z) is a homogeneous function of degree n in the variables x, y & z then

$$x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n.u$$

Problem

Verify Euler's theorem for the following function

1. u(x,y) = $x^2 - 2xy + y^2$
2. u(x,y) = $x^3 + y^3 + z^3 - 3xyz$

INCREASING AND DECREASING FUNCTION

Increasing function

A function y= f(x) is said to be an increasing function if $f(x_1) < f(x_2)$ for all $x_1 < x_2$.

The condition for the function to be increasing is that its first order derivative is always greater than zero .

i.e $\dfrac{dy}{dx} > 0$

Decreasing function

A function y= f(x) is said to be a decreasing function if $f(x_1) > f(x_2)$ for all $x_1 < x_2$.

The condition for the function to be decreasing is that its first order derivative is always less than zero .

i.e $\dfrac{dy}{dx} < 0$

Problems

1. Show that the function $y = x^3 + x$ is increasing for all x.
2. Find for what values of x is the function $y = 8 + 2x - x^2$ is increasing or decreasing ?

Maxima and Minima Function of a single variable

A function y = f(x) is said to have maximum at x = a if f(a) > f(x) in the neighborhood of the point x = a and f(a) is the maximum value of f(x) . The point x = a is also known as local maximum point.

A function y = f(x) is said to have minimum at x = a if f(a) < f(x) in the neighborhood of the point x = a and f(a) is the minimum value of f(x) . The point x = a is also known as local minimum point.

The points at which the function attains maximum or minimum are called the turning points or stationary points

A function y=f(x) can have more than one **maximum or minimum point** . Maximum of all the maximum points is called **Global maximum** and minimum of all the minimum points is called **Global minimum**.

A point at which neither maximum nor minimum is called **Saddle point**.

[Consider a function y = f(x). If the function increases upto a particular point x = a and then decreases it is said to have a maximum at x = a. If the function decreases upto a point x = b and then increases it is said to have a minimum at a point x=b.]

The necessary and the sufficient condition for the function y=f(x) to have a maximum or minimum can be tabulated as follows

	Maximum	**Minimum**
First order or necessary condition	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 0$
Second order or sufficient condition	$\frac{d^2y}{dx^2} < 0$	$\frac{d^2y}{dx^2} > 0$

Working Procedure

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

2. Equate $\frac{dy}{dx}$ =0 and solve for x. this will give the turning points of the function.

3. Consider a turning point x = a then substitute this value of x in $\frac{d^2y}{dx^2}$ and find the nature of the second derivative. If $\left(\frac{d^2y}{dx^2}\right)_{at\,x=a}$ < *0*, then the function has a maximum value at the point x = a. If $\left(\frac{d^2y}{dx^2}\right)_{at\,x=a}$ > *0*, then the function has a minimum value at the point x = a.

4. Then substitute x = a in the function y = f(x) that will give the maximum or minimum value of the function at x = a.

Problem

Find the maximum and minimum values of the following function

1. $y = x^3 - 3x + 1$